AIAA JOURNAL
Vol. 39, No. 11, November 2001

Stochastic Finite Element Analysis of Beam with Statistical Uncertainties

Ryohei Ishida*
Osaka Prefecture University, Osaka 599-8531, Japan

The eigenvalue problem of a beam with uncertain cross-sectional area is described. The cross-sectional area of the beam is expressed in a form containing stochastic variables, and the stochastic finite element method is applied to evaluate the influence of this uncertainty on the eigenvalue and the eigenvector. The formulation is carried out under the condition that the beam has a circular section. Numerical analysis is carried out with respect to the uniform beam and the optimal beam. The optimal beam has the maximum eigenvalue of first mode. The influence of the uncertainty of the beam cross-sectional area on the eigenvalue and the eigenvector is shown.

Nomenclature

A	=	cross-sectional area of beam
A_i	=	cross-sectional area in ith element
A_0	=	cross-sectional area of uniform beam
a_i	=	nondimensional cross-sectional area in ith element, A_i/A_0
d	=	autocorrelation parameter
E	=	Young's modulus
I	=	second moment of area
I_i	=	second moment of area in ith element
I_0	=	second moment of area of uniform beam
$[K]$	=	stiffness matrix
L	=	beam length
l	=	L/N
$[M]$	=	mass matrix
N	=	number of finite elements
r_i	=	nondimensional second moment of area in ith element, I_i/I_0
t	=	time
x	=	axial coordinate of beam
Y_i	=	nondimensional deflection at node i
y	=	deflection of beam
β	=	Δa
Δa	=	step parameter of nondimensional cross-sectional area
δ	=	variational symbol
δ_{ij}	=	Kronecker delta
ε_i	=	probabilistic variable ($-\varepsilon_{i\,\max} \le \varepsilon_i \le \varepsilon_{i\,\max}$)
$\varepsilon_{i\,\max}$	=	Δa for uniform beam and $\Delta a/\bar{a}_i$ for optimal beam
$\{\eta\}$	=	generalized deflection vector, eigenvector
θ_i	=	deflection angle at node i
λ	=	eigenvalue
ϱ	=	mass density of beam
ω	=	angular frequency
$\bar{}$	=	mean value, expectation

Superscripts

T	=	transpose
$0, i, ij$	=	zeroth-, first-, and second-order term of ε

I. Introduction

FOR structural design the finite element method (FEM) is a very powerful analysis tool. Usually, the structural analysis is applied under deterministic design conditions. However, most real structures are subjected to different conditions from their design conditions. Further, the elements of the structure or the structure itself will exhibit various types of uncertainty in the material properties or the dimensions because of their manufacturing process. These uncertainties will often induce unexpected significant influences on the desired functions of these structures. Therefore, it is important to evaluate the influence of such uncertainties in structures in a statistical manner. Such a statistical technique is important from the viewpoint of the reliability analysis of a structure and also the structural design based on a reliability analysis.

When we evaluate the influence on the structure of uncertainty statistically, the Monte Carlo simulation is often used. However, the Monte Carlo simulation requires a large number of randomly generated samples. So, the simulation seems not to be practical from the viewpoint of computation time when it is combined with the FEM. Therefore, the stochastic finite element method (SFEM) has been proposed and developed.[1,2] In the SFEM various uncertainties can be considered directly in the FEM formulation. Recently, SFEM has been applied to various fields, for example, to the transient heat-transfer problem[3] and dynamic analysis of the frame structure.[4] It has also been applied to sensitivity analyses to evaluate the influence of the random parameter or random boundary condition on the system response.[5]

Several studies[6–8] are highly relevant to the present work. Zhu and Wu[6] analyzed the eigenvalue problem of a beam with spatially varying material properties, namely, Young's modulus and mass density. The problem has been formulated by means of a method based on the local averages of the random vector field.[6] Ramu and Ganesan analyzed the eigenvalue problem of a beam column with material property uncertainty, random elastic support, and random loading.[7] They solved static response, free vibration, and stability problems. Deodatis and Graham applied the variability response functions to the plane stress problem with a stochastic elastic property and thickness and the eigenvalue problem of a cantilever beam with a stochastic elastic modulus and mass density.[8]

This paper deals with the stochastic finite element analysis of the eigenvalue problem for a beam structure with a circular cross section. To this aim, we first formulate the deterministic FEM for the beam. In the formulation of the SFEM, it is assumed that the beam has uncertainty in its transverse shape. The second-order perturbation procedure is introduced in the formulation of the SFEM. The SFEM analysis is carried out on both uniform and optimal beams. To adjust the optimal beam to the assumption on the present formulation, the optimization problem is formulated under the constant beam mass and is solved by using the constructive algorithm[9] based on the genetic algorithm.[10,11] In the formulation of the optimization problem, the stepped beam approximation is used. The optimal beam has the maximum eigenvalue of first mode. Numerical calculations are carried out to show the influences of the uncertainties on the transverse beam shape on the eigenvalues and eigenvectors. Four kinds of support condition of a beam are considered. The sensitivities and the coefficient of variation (C.O.V.) of the eigenvalues are illustrated. To calculate the C.O.V., it is assumed that the

Received 20 February 2001; revision received 18 May 2001; accepted for publication 21 May 2001.

*Assistant Professor, Department of Aerospace Engineering, 1-1, Gakuen-Cho, Sakai.

cross-sectional area is correlated between different finite elements. The 99.7% confidence intervals of the deflection functions are also shown.

II. Analysis

A. FEM Formulation

Consider an Euler–Bernoulli beam without internal damping. The strain energy U and the kinetic energy K are given as follows:

$$U = \frac{1}{2}\int_0^L EI\left(\frac{\partial^2 y}{\partial x^2}\right)^2 \mathrm{d}x, \qquad K = \frac{1}{2}\int_0^L \varrho A\left(\frac{\partial y}{\partial t}\right)^2 \mathrm{d}x \tag{1}$$

For the free-vibration problem Hamilton's variational principle is

$$\delta \int_{t_0}^{t_1} (U - K)\,\mathrm{d}t = 0 \tag{2}$$

Substituting Eq. (1) into Eq. (2) and considering $\delta y = 0$ at $t = t_0$ and $t = t_1$, we have

$$\int_{t_0}^{t_1}\int_0^L \varrho A \delta y \frac{\partial^2 y}{\partial t^2}\,\mathrm{d}x\,\mathrm{d}t + \delta \int_{t_0}^{t_1}\int_0^L \frac{1}{2} EI\left(\frac{\partial^2 y}{\partial x^2}\right)^2 \mathrm{d}x\,\mathrm{d}t = 0 \tag{3}$$

We discretize the beam into N small finite elements with equal length l and describe the integration in Eq. (3) as a summation of integration over each interval $l_{i-1} \le x \le l_i$ $(i = 1, 2, \ldots, N)$ and $l_i - l_{i-1} = l$. Assuming that the transverse shape of the beam is varying along its axis, the cross-sectional area and the second moment of area are functions of x. Denoting them as $A_i(x)$ and $I_i(x)$, respectively, Eq. (3) is rewritten as

$$\varrho \sum_{i=1}^N \int_{t_0}^{t_1}\int_{l_{i-1}}^{l_i} A_i(x)\delta y_i \frac{\partial^2 y_i}{\partial t^2}\,\mathrm{d}x\,\mathrm{d}t + E\sum_{i=1}^N \int_{t_0}^{t_1}\int_{l_{i-1}}^{l_i} I_i(x)\delta\left(\frac{\partial^2 y_i}{\partial x^2}\right)\left(\frac{\partial^2 y_i}{\partial x^2}\right)\mathrm{d}x\,\mathrm{d}t = 0 \tag{4}$$

Introducing the local coordinate system $\xi\,(0 \le \xi \le 1)$ for integration and the notation

$$x = l_{i-1} + l\xi, \qquad y_i(x,t) = lY_i(\xi)e^{j\omega t}, \qquad j = \sqrt{-1} \tag{5}$$

Eq. (3) is reduced to

$$\frac{\varrho L^4 \omega^2}{N^4}\sum_{i=1}^N \int_0^1 A_i(\xi)(\delta Y_i)Y_i\,\mathrm{d}\xi - E\sum_{i=1}^N \int_0^1 I_i(\xi)\delta\left(\frac{\partial^2 Y_i}{\partial \xi^2}\right)\left(\frac{\partial^2 Y_i}{\partial \xi^2}\right)\mathrm{d}\xi = 0 \tag{6}$$

Now, introducing the Hermite family of interpolation functions, we express the nondimensional deflection $Y_i(\xi)$ as follows:

$$Y_i(\xi) = \{H(\xi)\}^T\{\eta_i\} \tag{7}$$

where

$$\{H(\xi)\} = \{H_1(\xi) \quad H_2(\xi) \quad H_3(\xi) \quad H_4(\xi)\}^T$$
$$\{\eta_i\} = \{Y_{i-1} \quad \theta_{i-1} \quad Y_i \quad \theta_i\}^T \tag{8}$$

The functions $H_i(\xi)$ are the Hermite cubic interpolation functions[12] defined as

$$H_1(\xi) = 1 - 3\xi^2 + 3\xi^3, \qquad H_2(\xi) = \xi(1-\xi^2)$$
$$H_3(\xi) = 3\xi^2 - 2\xi^3, \qquad H_4(\xi) = \xi^2(\xi - 1) \tag{9}$$

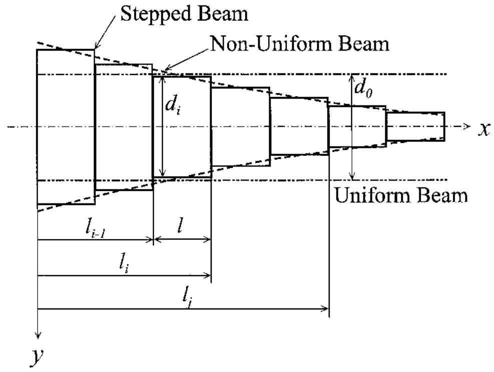

Fig. 1 Stepped beam approximation.

Substituting Eq. (7) into Eq. (6) and applying the variational principle, we have

$$\frac{\varrho A_0}{EI_0}\frac{L^4}{N^4}\omega^2 \sum_{i=1}^N [m_i]\{\eta_i\} - \sum_{i=1}^N [k_i]\{\eta_i\} = 0 \tag{10}$$

where the matrices $[m_i]$ and $[k_i]$ are as follows:

$$[m_i] = a_i \int_0^1 \{H(\xi)\}\{H(\xi)\}^T\,\mathrm{d}\xi$$
$$[k_i] = r_i \int_0^1 \{H''(\xi)\}\{H''(\xi)\}^T\,\mathrm{d}\xi \tag{11}$$

In the derivation of the two matrices of Eq. (11), the stepped beam approximation illustrated in Fig. 1 is used. In Eq. (11) the prime means the differential operation with respect to ξ. The full expressions for the matrices $[m_i]$ and $[k_i]$ are easily obtained and therefore are omitted here.

Defining the eigenvalue λ as

$$\lambda = (\varrho A_0/EI_0)(L^4/N^4)\omega^2 \tag{12}$$

Eq. (10) gives the simultaneous linear equation system

$$[K(r)]\{\eta\} - \lambda[M(a)]\{\eta\} = \{0\} \tag{13}$$

where $[K(r)]$ and $[M(a)]$ are the function of r_i and a_i, respectively. From the condition that Eq. (13) has a nontrivial solution $\{\eta\}$, we have the characteristic equation

$$|[K(r)] - \lambda[M(a)]| = 0 \tag{14}$$

B. Stochastic FEM Formulation

Now, introducing the small probabilistic variable ε, we express A_i as follows:

$$A_i = (1 + \varepsilon_i)\bar{A}_i \tag{15}$$

Denoting the expectation of a_i as $\bar{a}_i (= \bar{A}_i/A_0)$, a_i and r_i are expressed as

$$a_i = (1+\varepsilon_i)\bar{a}_i, \qquad r_i = (1+\varepsilon_i)^2\bar{a}_i^2 \tag{16}$$

Referring to Eq. (16), we express the mass and the stiffness matrices as follows:

$$[M] = [M^0] + \sum_{i=1}^N [M^i]\varepsilon_i \tag{17}$$

$$[K] = [K^0] + 2\sum_{i=1}^N [K^i]\varepsilon_i + \sum_{j=1}^N\sum_{k=1}^N [K^{jk}]\varepsilon_j\varepsilon_k\delta_{jk} \tag{18}$$

The matrices $[M^0]$ and $[K^0]$ are the mass matrix and the stiffness matrix replacing a_i and r_i by $\bar{a}_i$ and $\bar{r}_i$ in Eq. (11), respectively. The detailed forms of the other matrices are omitted.

Further, referring to Eqs. (17) and (18), we express the eigenvalue λ and the eigenvector $\{\eta\}$ as

$$\lambda = \lambda^0 + \sum_{i=1}^{N} \lambda^i \varepsilon_i + \frac{1}{2}\sum_{j=1}^{N}\sum_{k=1}^{N} \lambda^{jk}\varepsilon_j\varepsilon_k + \cdots \qquad (19)$$

$$\{\eta\} = \{\eta^0\} + \sum_{i=1}^{N}\{\eta^i\}\varepsilon_i + \frac{1}{2}\sum_{j=1}^{N}\sum_{k=1}^{N}\{\eta^{jk}\}\varepsilon_j\varepsilon_k + \cdots \qquad (20)$$

Substituting Eqs. (17–20) into Eq. (14), collecting the terms of equal order with respect to the parameter ε we have the following zeroth-, first-, and second-order linear equation system for the eigenproblem.

Zeroth order:

$$(K^0 - \lambda^0 M^0)\{\eta^0\} = 0 \qquad (21)$$

First order:

$$(K^0 - \lambda^0 M^0)\{\eta^i\} = -(2K^i - \lambda^0 M^i - \lambda^i M^0)\{\eta^0\} \qquad (22)$$

Second order:

$$(K^0 - \lambda^0 M^0)\{\eta^{jk}\} = -(2K^j - \lambda^0 M^j - \lambda^j M^0)\{\eta^k\}$$
$$-(2K^k - \lambda^0 M^k - \lambda^k M^0)\{\eta^j\}$$
$$-\left(2\delta_{jk}K^{jk} - \lambda^j M^k - \lambda^k M^j - \lambda^{jk}M^0\right)\{\eta^0\} \qquad (23)$$

In the preceding system the square brackets ([]) as a symbol of matrices are omitted.

The first derivative of eigenvalue λ^l is

$$\lambda^l = \frac{\{\eta^0\}^T(2K^l - \lambda^0 M^l)\{\eta^0\}}{\{\eta^0\}^T M^0\{\eta^0\}} \qquad (24)$$

and the first derivative of eigenvector $\{\eta^l\}$ is calculated from

$$\begin{bmatrix} K^0 - \lambda^0 M^0 \\ 2\{\eta^0\}^T M^0 \end{bmatrix}\{\eta^l\} = -\begin{bmatrix} 2K^l - \lambda^l M^0 - \lambda^0 M^l \\ \{\eta^0\}^T M^l \end{bmatrix}\{\eta^0\} \qquad (25)$$

Obviously, the matrices of both sides of the preceding equation are not square. Now, denote the matrices of the left- and the right-hand sides of Eq. (25) as A and B, respectively. Multiplying both sides by A^T, we obtain

$$\{\eta^l\} = -(A^T A)^{-1}(A^T B)\{\eta^0\} \qquad (26)$$

The second derivatives for the eigenvalue and the eigenvector, namely, λ^{jk} and $\{\eta^{jk}\}$, are derived in the same manner.

The expectation and the variant for the eigenvalue and the eigenvector are derived by using the first and second derivatives for the eigenvalue and the eigenvector. Assuming that ε is a random quantity with expectation zero, the expectation and the variant are obtained from the following expressions.

For eigenvalue:

$$E[\lambda] = \lambda^0 + \frac{1}{2}\sum_{k=1}^{N}\sum_{l=1}^{N}\lambda^{kl}E[\varepsilon_k\varepsilon_l] \qquad (27)$$

$$\mathrm{Var}[\lambda] = \sum_{j=1}^{N}\sum_{k=1}^{N}\lambda^j\lambda^k E[\varepsilon_j\varepsilon_k]$$
$$+\frac{1}{4}\sum_{j=1}^{N}\sum_{k=1}^{N}\sum_{l=1}^{N}\sum_{m=1}^{N}\lambda^{jk}\lambda^{lm}(E[\varepsilon_j\varepsilon_k\varepsilon_l\varepsilon_m] - E[\varepsilon_j\varepsilon_k]E[\varepsilon_l\varepsilon_m]) \qquad (28)$$

For eigenvector:

$$E[\{\eta\}] = \{\eta^0\} + \frac{1}{2}\sum_{k=1}^{N}\sum_{l=1}^{N}\{\eta^{kl}\}E[\varepsilon_k\varepsilon_l] \qquad (29)$$

$$\mathrm{Var}[\{\eta\}] = \sum_{j=1}^{N}\sum_{k=1}^{N}\mathrm{diag}[\{\eta^j\}]\{\eta^k\}E[\varepsilon_j\varepsilon_k]$$
$$+\frac{1}{4}\sum_{j=1}^{N}\sum_{k=1}^{N}\sum_{l=1}^{N}\sum_{m=1}^{N}\mathrm{diag}[\{\eta^{jk}\}]\{\eta^{lm}\}(E[\varepsilon_j\varepsilon_k\varepsilon_l\varepsilon_m]$$
$$- E[\varepsilon_j\varepsilon_k]E[\varepsilon_l\varepsilon_m]) \qquad (30)$$

where diag $[\{\eta\}]$ is a diagonal matrix with component of vector $\{\eta\}$ as its diagonal component.

III. Numerical Results and Discussion

The numerical calculation has been carried out under the preceding formulations. Comparing the numerical results by the SFEM with the results by the deterministic FEM for various small perturbation parameter ε, the validity of the stochastic finite element formulation was checked. The range of the parameter ε has also been determined simultaneously from the preceding calculation. We consider the four kinds of support condition of beam, which are one clamped end and the other free (C/F), clamped and simply supported (C/S), clamped and clamped (C/C), and simply supported and simply supported (S/S).

The beam optimization problem is formulated in the Appendix. The optimal beam has the maximum eigenvalue of the first mode for each support conditions of the beam. The optimization problem is solved under the condition that $a_{\min} = 0.1$ (corresponding minimum nondimensional diameter of the beam is 0.316), $\Delta a = 0.025$, and $N = 16$. $a_{\min}$ and Δa are defined in the Appendix. Figure 2

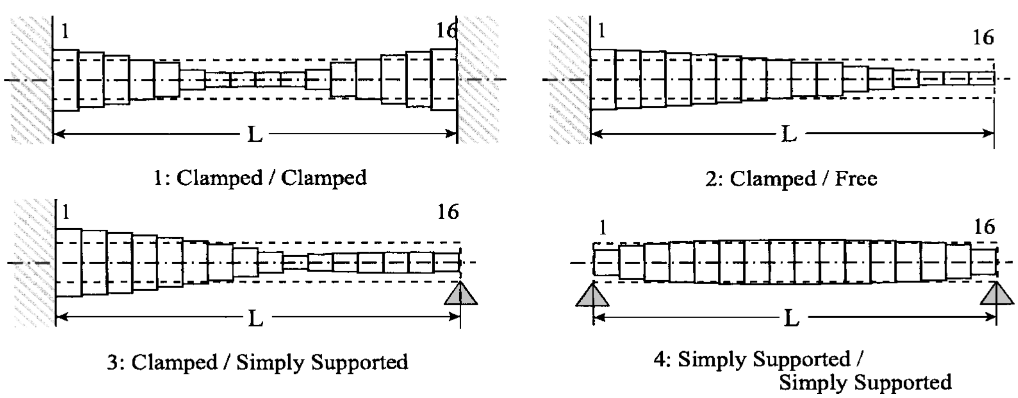

Fig. 2 Optimal beam shape.

shows the optimal stepped beam shape. In the figures the dotted line shows the uniform beam. Table 1 shows the eigenvalue of the first mode of the present problem; $\lambda_{1\mathrm{uni}}$ and $\lambda_{1\mathrm{opt}}$ show the eigenvalues of the first mode for the optimal beam and the uniform one, respectively.

To calculate the expectation, etc., we assume that the variation of the nondimensional cross-sectional area is expressed as $a_i = \bar{a}_i + \beta$ in which the parameter β varies in the range $[-\Delta a, +\Delta a]$. The quantity Δa is defined as the step parameter of a nondimensional cross-sectional area in the Appendix. For the uniform beam the expectation of non-cross-sectional area is unity, namely, $\bar{a}_i = 1$ for all finite elements. Thus, from the definition of Eq. (16), the range of ε is $[-\Delta a, +\Delta a]$. On the other hand, for the optimal beam, ε_i varies in the range $[-\Delta a/\bar{a}_i, +\Delta a/\bar{a}_i]$.

Figures 3a–3d show the variation of the first eigenvalue λ_1 vs the variation of β for the uniform beam for each support condition. Figures 4a–4d are for the optimal beam. For a C/C uniform beam we can see that λ_1 is very sensitive to the variation of β near both clamped ends. On the other hand, for the C/C optimal beam, λ_1 is sensitive to β near the center of the beam and is not sensitive near both clamped ends. For a C/F uniform beam, λ_1 varies gradually to the variation of β. For the optimal C/F beam the eigenvalue λ_1 varies largely to the variation of the parameter β near the free end. For C/S and S/S uniform beams the variation of λ_1 is quite gradual for the uniform and optimal beams. Especially, the S/S beam is insensitive to the variation of β. From these figures we can see that λ_1 is sensitive to the variation of β in every element for the uniform beam. On the other hand, for the optimal beam the variation of λ_1 is very sensitive to the local variation of β, for example, near the center of the C/C optimal beam and near the free end of the C/F optimal beam.

Table 1 Eigenvalue of uniform beam and optimal beam

Eigenvalue	C/C	C/F	C/S	S/S
$\lambda_{1\mathrm{uni}}$	500.56	12.36	237.70	97.39
$\lambda_{1\mathrm{opt}}$	2610.95	131.99	495.00	109.18

In the preceding discussion we have seen the sensitivity of the λ_1 to β in each element. To calculate the expectation and the variant from Eqs. (27–30), we define a correlation function as follows:

$$E[\varepsilon_i \varepsilon_j] = R_\varepsilon(l_i - l_j) = c_{ij} \exp(-d|l_i - l_j|/L) \tag{31}$$

The correlation distance of the field is defined as $2L/d$ (Ref. 6); c_{ij} is the coefficient with respect to variation. In this paper we define c_{ij} as

$$c_{ij} = \varepsilon_{i\,\max}\varepsilon_{j\,\max} \tag{32}$$

Thus, Eq. (31) is rewritten as

$$E[\varepsilon_i \varepsilon_j] = \Delta a^2 \exp(-d|l_i - l_j|/L) \tag{33}$$

for a uniform beam and

$$E[\varepsilon_i \varepsilon_j] = \Delta a^2 \exp(-d|l_i - l_j|/L)/(\bar{a}_i \bar{a}_j) \tag{34}$$

for an optimal beam. The fourth-order moment of ε is written by using the second moment $E[\varepsilon_i \varepsilon_j]$ as follows:

$$E[\varepsilon_i \varepsilon_j \varepsilon_k \varepsilon_l] = E[\varepsilon_i \varepsilon_j]E[\varepsilon_k \varepsilon_l] + E[\varepsilon_i \varepsilon_k]E[\varepsilon_j \varepsilon_l] + E[\varepsilon_i \varepsilon_l]E[\varepsilon_j \varepsilon_k] \tag{35}$$

Odd-order moments are zero because the expectation of ε is zero.

Tables 2 and 3 show the C.O.V. for various values of the parameter d. C.O.V.s are defined as

$$\text{C.O.V.} = \sqrt{\mathrm{Var}(\lambda_1)/(E[\lambda_1])^2} \tag{36}$$

These are also shown in Figs. 5 and 6. Table 2 and Fig. 5 are the result of the uniform beam, and Table 3 and Fig. 6 are that of the optimal beam. In these tables and figures, C.O.V.s are standardized by Δa.

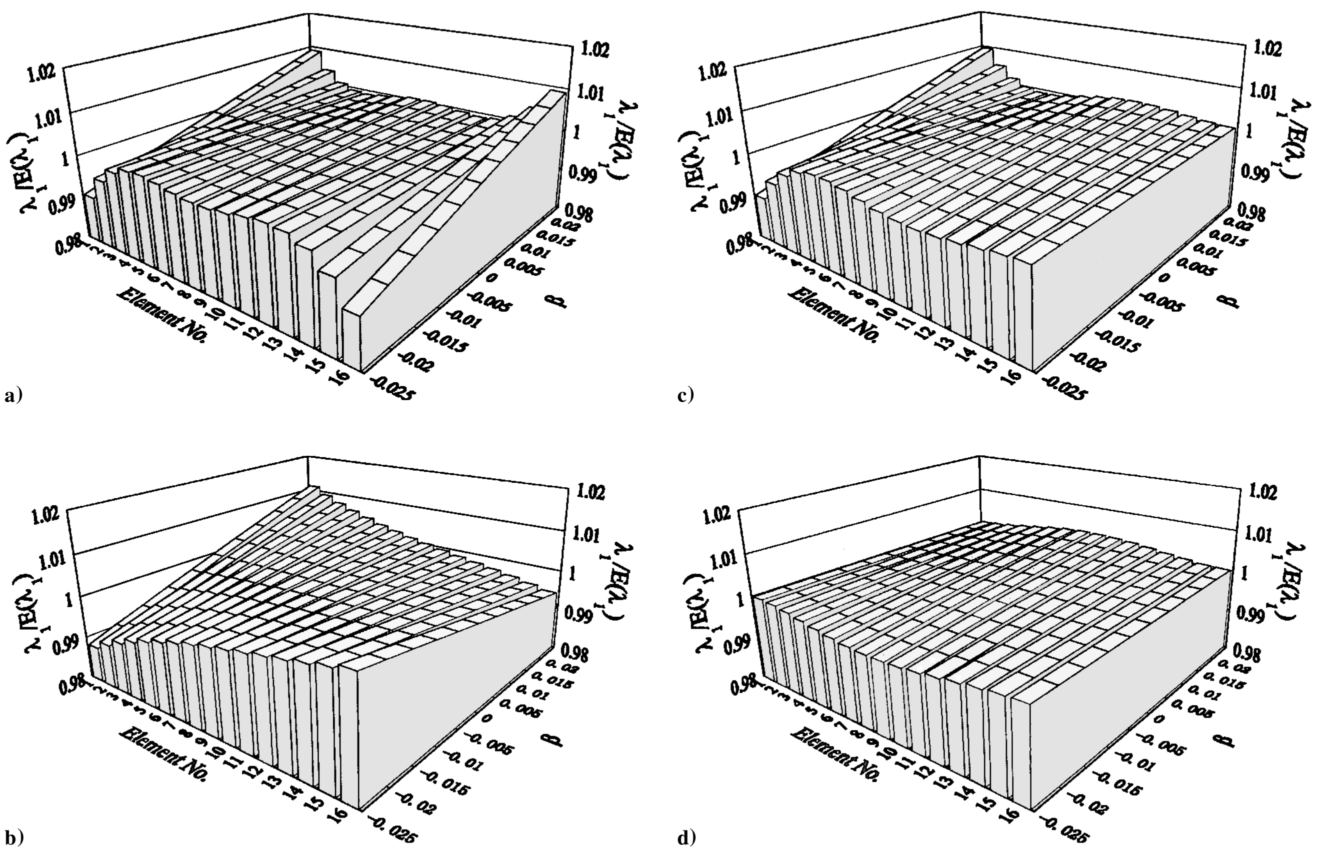

Fig. 3 Variation of eigenvalue for a) C/C, b) C/F, c) C/S, and d) S/S uniform beams.

From the tables and figures it can be seen that, as the parameter d increases, the coefficient monotonously decreases and asymptotically approaches the values of the case of no correlation ($d = \infty$). In the case of the uniform C/F beam, the coefficient becomes maximum at $d \approx 3$. For the optimal C/C and C/F beams the coefficient is larger in comparison with the values for the uniform beams of the same support conditions. On the contrary, for the optimal C/S and S/S beams the coefficient is smaller than the values for the uniform C/S and S/S beams. This is understood from the fact that optimal C/C and C/F beams are very sensitive to the local variation of β (Fig. 4).

Figures 7 and 8 show the expectation and the 99.7% confidence interval of the deflection function y. These results are calculated under the condition that $\Delta a = 0.025$ and $d = \infty$ (Gaussian process). The deflection function is normalized to be convenient for comparison, where the 99.7% confidence interval means that the probability is 99.7% for a measurement of y to lie between $E(y) - 3\sigma_y$ and $E(y) + 3\sigma_y$. So, in Figs. 7 and 8 the interval is referred to as the "3σ region." For the uniform beam the interval is very narrow for all support conditions. This means that the influences of the stochastic variable ε on the deflection function are very weak. On the contrary, the optimal beams, except the S/S beam, have a very wide 99.7% confidence interval. It seems that, for the optimal beam, the

Table 2 Coefficient of variation for various *d* (uniform beam)

d	C/C	C/F	C/S	S/S
0.0	1.0000	1.0000	1.0000	1.0000
5.0	0.7019	1.4960	0.6688	0.6587
10.0	0.6781	1.2744	0.6058	0.5151
15.0	0.6548	1.1297	0.5679	0.4401
20.0	0.6350	1.0380	0.5408	0.3956
∞	0.5775	0.8422	0.4735	0.3055

Table 3 Coefficient of variation for various *d* (optimal beam)

d	C/C	C/F	C/S	S/S
0.0	2.1375	3.0707	0.8162	0.9971
5.0	2.4073	3.8409	0.4649	0.5553
10.0	2.2395	3.6897	0.3752	0.4208
15.0	2.0681	3.5363	0.3337	0.3575
20.0	1.9377	3.4165	0.3097	0.3212
∞	1.6123	3.0865	0.2601	0.2495

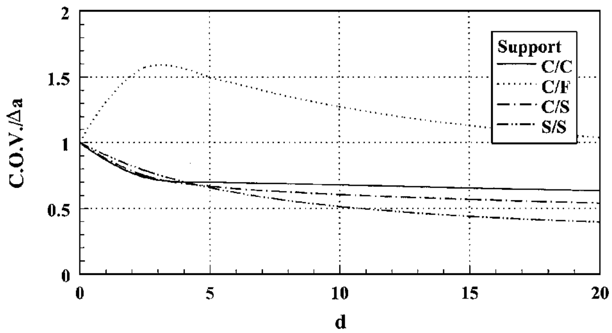

Fig. 5 Coefficient of variation for various *d* (uniform beam).

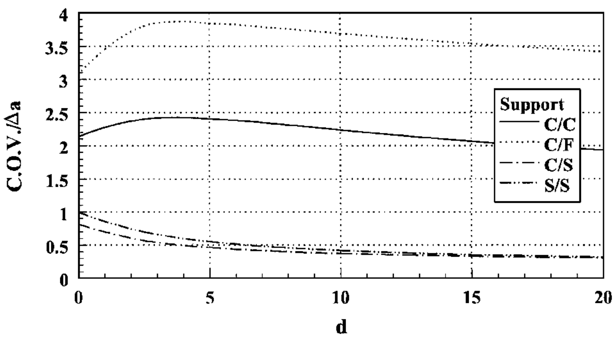

Fig. 6 Coefficient of variation for various *d* (optimal beam).

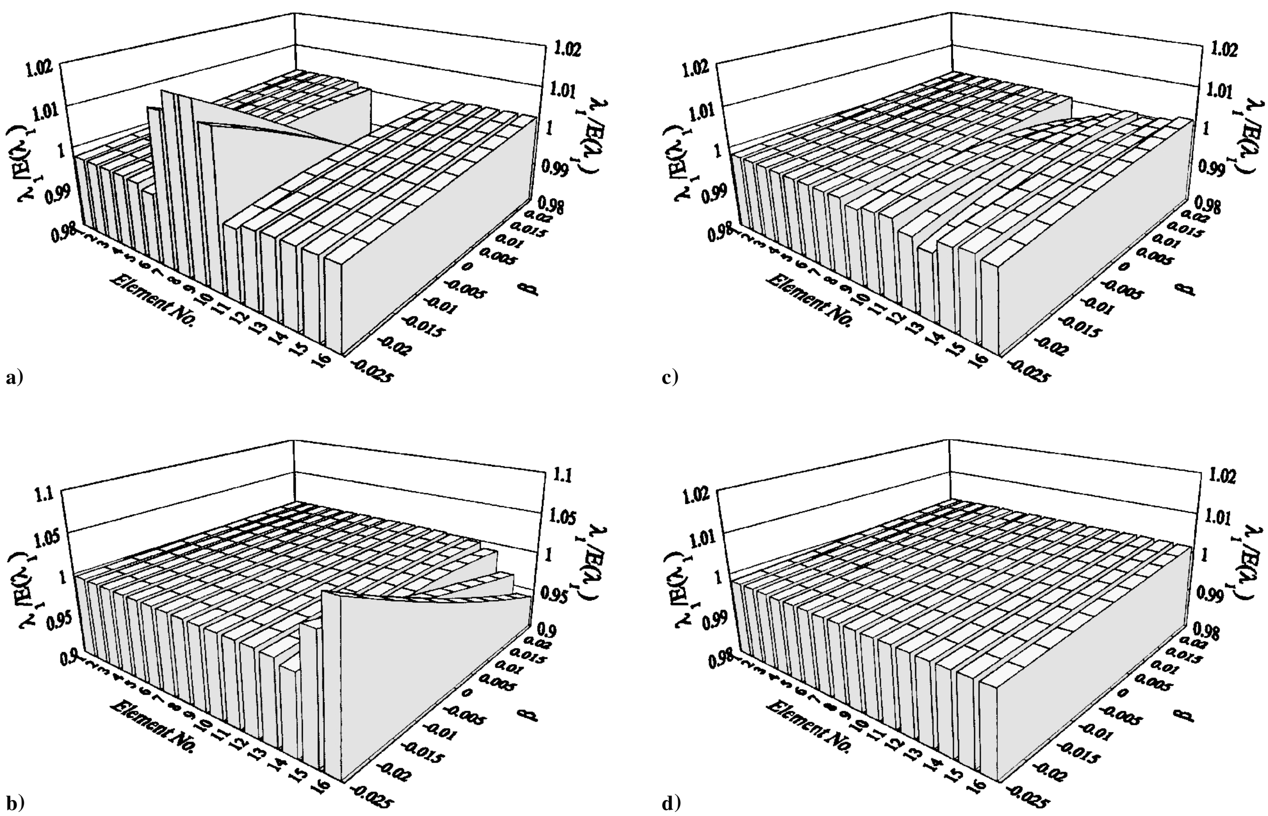

Fig. 4 Variation of eigenvalue for a) C/C, b) C/F, c) C/S, and d) S/S optimal beams.

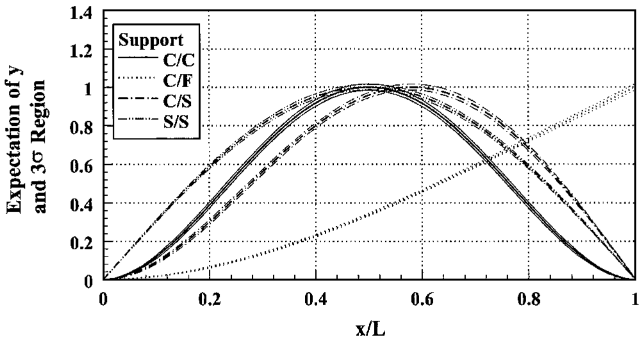

Fig. 7 Expectation of deflection y and 99.7% confidence interval for uniform beams.

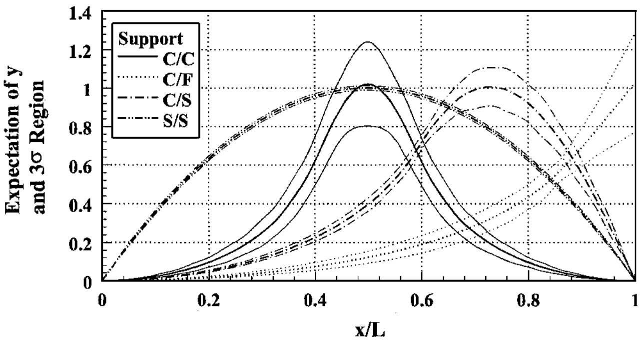

Fig. 8 Expectation of deflection y and 99.7% confidence interval for optimal beams.

stochastic variable will have a significant influence on the deflection of the beam.

IV. Conclusions

In this paper the stochastic finite element method formulation is performed on the assumption that the beam has uncertainty in its transverse shape. The formulation is performed with respect to uniform beams and optimal beams. We assumed that the cross-sectional area of the beams has stochastic variation. Numerical calculation is carried out to evaluate the influences of the stochastic variation of the cross-sectional area to the variation of the first eigenvalue. Four kinds of support conditions, namely C/C, C/F, C/S, and S/S, are considered. Assuming the correlation function, the influence of the parameter related to the correlation distance to the coefficient of variation is obtained. The 99.7% confidence intervals of the deflection function are also shown for the uniform and optimal beams.

Appendix: Optimization Problem of Beam

In this Appendix we formulate the transverse shape optimization problem. The problem is formulated as finding the transverse shape that has the maximum eigenvalue of first mode under the constant beam mass. The objective function is the eigenvalue of first mode. In our problem the parameter to be optimized is the nondimensional cross-sectional area a_i as the parameter governing the transverse shape. So, the optimization problem is written as finding the distribution of a_i to maximize the objective function.

In this paper the cross section of the beam has a circular cross section; the relation between the just-mentioned nondimensional cross-sectional area a_i and the nondimensional second moment of area r_i is

$$r_i = a_i^2 \tag{A1}$$

Futher, introducing $a_{\min}$ as the minimum value of a_i to avoid the zero cross-sectional area, we express the nondimensional cross-sectional area a_i as

$$a_i = a_{\min} + m_i \Delta a \tag{A2}$$

where m_i are integers.

The beam mass is expressed as

$$\varrho A_0 l \sum_{i=1}^{N} a_i$$

for the stepped beam. For the uniform beam appearently $a_i = 1$, and so the mass of the uniform beam is $\varrho A_0 l N = \varrho A_0 L$. To prescribe the constant beam mass condition, the following equation should be satisfied:

$$\sum_{i=1}^{N} a_i = N \tag{A3}$$

From Eqs. (38) and (39) we have

$$\sum_{i=1}^{N} m_i = \frac{N(1 - a_{\min})}{\Delta a} = M \tag{A4}$$

M is the total number of the parameters Δa. Once $a_{\min}$ and Δa are fixed, M is determined from Eq. (40). So the optimization problem is rewritten as a problem to find the best combination of m_i with the maximum eigenvalue of first mode under constraint that

$$\sum_{i=1}^{N} m_i = M$$

In this optimization problem there are a great number of combinations of m_i that satisfy the constraint. Actually, the number of the possible combinations of m_i is given by $(M + N - 1)!/\{M!(N-1)!\}$. For example, taking N, $a_{\min}$, and Δa as 16, 0.1, and 0.05, respectively, then $M = 288$, and the number of the possible combination of m_i reaches $303!/(288! \times 15!) \approx 8.96 \times 10^{24}$. To find the best combination, we use the constructive algorithm.[9]

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A. Berman
Associate Editor